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# Twisted $p$-adic $(h, q)$-$L$-functions

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ABSTRACT

By using the $q$-Volkenborn integral on $\mathbb{Z}_p$, in Simsek (2006) [33] and Simsek (2007) [34], generating functions for the $(h, q)$-Bernoulli polynomials and numbers were defined. By using these functions, we define a new twisted $(h, q)$-partial zeta function which interpolates the twisted $(h, q)$-Bernoulli polynomials and generalized twisted $(h, q)$-Bernoulli numbers at negative integers. We give a relation between twisted $(h, q)$-partial zeta functions and the twisted $(h, q)$-two-variable $L$-function. We find the value of this function at $s = 0$. We also find the residue of this function at $s = 1$. We construct a $p$-adic twisted $(h, q)$-$L$-function which interpolates the twisted $(h, q)$-Bernoulli polynomials:

$$L^{(h)}_{\xi,p,q}(1-n,t,\chi) = -\frac{B^{(h)}_{n,\chi_n,\xi}(p^*t,q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t,q^p)}{n}.$$

Furthermore, we construct an integral representation of the twisted $(h, q)$-two-variable $L$-function. We give some applications related to the $p$-adic twisted $(h, q)$-$L$-function and the twisted $(h, q)$-Bernoulli polynomials.



## 1. Introduction, definitions and notation

The history of the Bernoulli numbers goes back to Bernoulli in the 16th century. From Bernoulli's time to this, these numbers have been defined in many different ways. Thus many applications of these numbers and their generating functions have been looked for by many authors in the literature. Many different functions are used to obtain generating functions for the Bernoulli numbers. In this paper, we study other kinds of Bernoulli numbers and their interpolation function (cf. [1–44]; see also the references cited in each of these earlier works).

In [12], Kim constructed $p$-adic $q$-$L$-functions. He gave fundamental properties of these functions. By using the $p$-adic $q$-integral, he also constructed a generating function for Carlitz's $q$-Bernoulli number. Recently, Fox [5], Young [44] and Kim [17] (see also the references cited in each of these earlier works) studied two-variable $p$-adic $L$-functions related to the Dirichlet character. In this paper we construct a two-variable *twisted* $p$-adic $L$-function, which is given in Section 3.

Throughout this paper we use the following notation:

Let $p$ be an odd prime. Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of integers, the field of rational numbers, the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ denote the $p$-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ is denoted by $|\cdot|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. The integer $p^*$ is defined by

$$p^* = \begin{cases} p & \text{if } p > 2, \\ 4 & \text{if } p = 2. \end{cases}$$

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Let $w$ denote the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character $\chi$, we define $\chi_n = \chi w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. Let

$$\mathcal{D} = \left\{ s \in \mathbb{C}_p : |s|_p \leq |p^*|^{-1} p^{-\frac{1}{p-1}} \right\} \quad \text{and} \quad |p^*|^{-1} p^{-\frac{1}{p-1}} > 1$$

(see for details [7,22,41,18,17,44]; see also the references cited in each of these earlier works).

Let $\chi$ be a primitive Dirichlet character with conductor $f$. The Dirichlet $L$-function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1 \tag{1.1}$$

(cf. [4,1,5,3,7,16,17,12,24,23,34,35,37,41,43,44]; see also the references cited in each of these earlier works). The function $L(s, \chi)$ is analytically continued to the complex $s$-plane; we have

$$L(1-n, \chi) = -\frac{B_{n,\chi}}{n}, \tag{1.2}$$

where $n \in \mathbb{Z}^+$ ($\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$), and $B_{n,\chi}$, the usual generalized Bernoulli numbers, are defined by

$$\sum_{a=0}^{f-1} \frac{\chi(a) e^{at} t}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} \quad \text{(cf. [1–44])}.$$

We recall [7,44,24] that the $p$-adic analogue of (1.2) is the Kubota–Leopoldt $p$-adic $L$-function $L_p(s, \chi)$, which is a unique analytic function on $\mathcal{D}$ (except for a simple pole at $s = 1$ when $\chi \equiv 1$) for which

$$L_p(1-n, \chi) = -\frac{(1 - \chi_n(p) p^{n-1}) B_{n,\chi_n}}{n},$$

where $n \in \mathbb{Z}^+$ and $\chi_n$ denotes the Dirichlet character $\chi w^{-n}$ (for details see [4,3,7,22,41,28,23,12,10,18]; see also the references cited in each of these earlier works).

Kim et al. [20] defined twisted $q$-Bernoulli numbers by using $p$-adic invariant integrals on $\mathbb{Z}_p$. They gave twisted $q$-zeta functions and $q$-$L$-series which interpolate twisted $q$-Bernoulli numbers. In [32], the author constructed generating functions for $q$-generalized Euler numbers and polynomials. The author also constructed a complex analytic twisted $l$-series, which interpolates twisted $q$-Euler numbers at non-positive integers. In [31,30], the author gave analytic properties of twisted $L$-functions. The author defined twisted Bernoulli polynomials and numbers. The author gave a relation between twisted Bernoulli numbers and twisted $L$-functions. The author also gave $q$-analogues of these numbers and functions. Young [44] defined some $p$-adic integral representations for the two-variable $p$-adic $L$-function. For powers of the Teichmüller character, he used the integral representation to extend the $L$-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials. In [16], by using $q$-Volkenborn integration, Kim constructed a new $(h, q)$-extension of the Bernoulli numbers and polynomials. He defined $(h, q)$-extensions of the zeta functions which interpolated new $(h, q)$-extensions of the Bernoulli numbers and polynomials. In [33], the author defined twisted $(h, q)$-Bernoulli numbers, zeta functions and $L$-functions. The author also gave relations between these functions and numbers. In [19], Kim and Rim constructed a two-variable $L$-function, which interpolates the generalized Bernoulli polynomials associated with $\chi$. By means of Mellin transforms, they gave a complex integral representation for the two-variable Dirichlet $L$-function. They also found some properties of the two-variable Dirichlet $L$-function. In [17], Kim constructed a two-variable $p$-adic $q$-$L$-function which interpolates the generalized $q$-Bernoulli polynomials associated with the Dirichlet character. He also gave a $p$-adic integral representation for this two-variable $p$-adic $q$-$L$-function and derived a $q$-extension of the generalized formula of Diamond, Ferrero and Greenberg for the two-variable $p$-adic $L$-function in terms of the $p$-adic gamma and log gamma functions. In [36], Simsek et al. defined a $q$-analogue two-variable $L$-function.

In [18], Kim constructed a new $q$-extension of generalized Bernoulli polynomials related to $\chi$ deriving from his work [16], and derived the existence of a specific $p$-adic interpolation function which interpolate the $q$-extension of generalized Bernoulli polynomials at negative integers. He gave the values of the partial derivatives of this function.

Throughout of this paper, we shall use the following standard notation:

$$[x]_q = \begin{cases} \dfrac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases} \quad \text{(cf. [8,13,14,21,36,1])}.$$

When one talks of a $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, and a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we assume $|1-q|_p < p^{-\frac{1}{p-1}}$, so $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we assume $|q| < 1$; cf. [1–41].

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p$ is a uniformly differentiable function$\}$, the $p$-adic $q$-integral ($q$-Volkenborn integral) is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x), \tag{1.3}$$

where $\mu_q$ denotes the $p$-adic $q$-Haar distribution which was originally introduced by Kim [10]:

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{q^a}{[p^N]_q}, \quad N \in \mathbb{Z}^+.$$

If $q \to 1$ in (1.3), then we have

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \quad \text{(cf. [40,45,13])}. \tag{1.4}$$

Observe that in (1.4), we readily see that $I_1(f) = \lim_{q\to 1} I_q(f)$.

If we take $f_1(x) = f(x+1)$ in (1.4), then we have

$$I_1(f_1) = I_1(f) + f'(0), \tag{1.5}$$

where $f'(0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)\,|_{x=0}$; cf. [15].

The $p$-adic $q$-integral is used not only in mathematics but also in mathematical physics, in the theory of $q$-Stirling numbers, $q$-Mahler theory and other areas (cf. [40,45,22,13,37,41]; see also the references cited in each of these earlier works).

Let $p$ be a fixed prime. For a fixed positive integer $f$ with $(p, f) = 1$, we set (see [13])

$$\mathbb{X} = \mathbb{X}_f = \lim_{\overleftarrow{N}} \mathbb{Z}/fp^N\mathbb{Z},$$

$$\mathbb{X}_1 = \mathbb{Z}_p,$$

$$\mathbb{X}^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} \cup a + fp\mathbb{Z}_p$$

and

$$a + fp^N\mathbb{Z}_p = \left\{x \in \mathbb{X} \mid x \equiv a (\mathrm{mod}\, fp^N)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < fp^N$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_1(x) = \int_{\mathbb{X}} f(x) \mathrm{d}\mu_1(x) \quad \text{(cf. [14])}. \tag{1.6}$$

By (1.5), we readily see that

$$I_1(f_b) = I_1(f) + \sum_{j=0}^{b-1} f'(j) \quad \text{(cf. [14])} \tag{1.7}$$

where $f_b(x) = f(x + b)$, $b \in \mathbb{Z}^+$. There are various applications of the $p$-adic $q$-Volkenborn integral. For example, by using this integral, generating functions for the Bernoulli, Euler and Genocchi numbers are defined (cf. [40,45,22,13,14,16,37,33]; see also the references cited in each of these earlier works).

According to [42,29], for each integer $N \ge 0$, $C_{p^N}$ denotes the multiplicative group of the primitive $P^N$th roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$. Let

$$\mathbb{T}_p = \left\{\xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \ge 0\right\} = \bigcup_{N\ge 0} C_{p^N}.$$

The dual of $\mathbb{Z}_p$, in the sense of $p$-adic Pontrjagin duality, is $\mathbb{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups $C_{p^N}$ of order $p^N$ with $N \ge 0$, with the discrete topology. $\mathbb{T}_p$ admits a natural $\mathbb{Z}_p$-module structure which we shall write exponentially, namely as $\xi^x$ for $\xi \in \mathbb{T}_p$ and $x \in \mathbb{Z}_p$. $\mathbb{T}_p$ can be embedded discretely in $\mathbb{C}_p$ as the multiplicative $p$-torsion subgroup, and we choose, once and for all, one such embedding. If $\xi \in \mathbb{T}_p$, then $\phi_\xi : (\mathbb{Z}_p, +) \to (\mathbb{C}_p, .)$ is the locally constant character, $x \to \xi^x$, which is a locally analytic character if $\xi \in \{\xi \in \mathbb{C}_p : v_p(\xi - 1) > 0\}$. Then $\phi_\xi$ has a continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p, .)$ (cf. [42,29,9,21]; see also the references cited in each of these earlier works).

If $\xi \in \mathbb{C}$, then we assume that $\xi$ is an $r$th root of 1 with $r \in \mathbb{Z}^+$, the set of positive integers.

By using the $p$-adic $q$-Volkenborn integral, the author [33] constructed a generating function for *the twisted* $(h, q)$-*extension of the Bernoulli numbers* $B_{n,\xi}^{(h)}(q)$ by means of the following generating function:

$$F_{\xi,q}^{(h)}(t) = \frac{\log q^h + t}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q)\frac{t^n}{n!}, \quad |t + \log(\xi q^h)| < 2\pi, \tag{1.8}$$

where $\xi$ is an $r$th root of 1. By applying the *umbral calculus* convention in the above equation, and the usual convention of symbolically replacing $(B_{\xi}^{(h)}(q))^n$ by $B_{n,\xi}^{(h)}(q)$, we then have

$$B_{0,\xi}^{(h)}(q) = \frac{\log q^h}{\xi q^h - 1} \tag{1.9}$$
$$\xi q^h (B_{\xi}^{(h)}(q) + 1)^n - B_{n,\xi}^{(h)}(q) = \delta_{1,n}, \quad n \geq 1,$$

where $\delta_{1,n}$ is the Kronecker symbol.

Observe that if $\xi = 1$; then (1.8) reduces to the following generating function:

$$\begin{aligned} F_{\xi,q}^{(h)}(t)\mid_{\xi=1} &= F_{q}^{(h)}(t) \\ &= \frac{\log q^h + t}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,1}^{(h)}(q)\frac{t^n}{n!} \quad \text{(cf. [16])} \end{aligned}$$

where $B_{n,1}^{(h)}(q) = B_n^{(h)}(q)$. If $q \to 1$ in the above, then we have

$$F(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi$$

where $B_n$ denotes the usual Bernoulli numbers (see [1–44]).

We note that the twisted $(h, q)$-extensions of Bernoulli numbers are related to the Frobenius–Euler numbers, which are given as follows:

Let $u$ be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius–Euler numbers $H_n(u)$ belonging to $u$ are defined by means of the following generating function (see [26,28,37]):

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}.$$

By using the above equation, and following the usual convention of symbolically replacing $H^n(u)$ by $H_n(u)$, we have

$$H_0 = 1 \quad \text{and} \quad (H(u) + 1)^n = uH_n(u) \quad \text{for } (n \geq 1).$$

We also note that

$$H_n(-1) = E_n,$$

where $E_n$ denotes the classical Euler numbers (see [26,28,37]).

We now give a relation between the twisted $(h, q)$-extensions of Bernoulli numbers and the Frobenius–Euler numbers as follows:

By (1.8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q)\frac{t^n}{n!} &= \left(\frac{\log q^h}{\xi q^h - 1}\right)\left(\frac{1 - \xi^{-1}q^{-h}}{e^t - \xi^{-1}q^{-h}}\right) + \left(\frac{t}{\xi q^h - 1}\right)\left(\frac{1 - \xi^{-1}q^{-h}}{e^t - \xi^{-1}q^{-h}}\right) \\ &= \left(\frac{1}{\xi q^h - 1}\right)\sum_{n=0}^{\infty}\left((\log q^h)H_n(\xi^{-1}q^{-h}) + nH_{n-1}(\xi^{-1}q^{-h})\right)\frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on either side of the above, we easily obtain

$$B_{n,\xi}^{(h)}(q) = \frac{(\log q^h)H_n(\xi^{-1}q^{-h}) + nH_{n-1}(\xi^{-1}q^{-h})}{\xi q^h - 1}. \tag{1.10}$$

Substituting $\xi^r = 1$, $\xi \neq 1$ and $q \to 1$ into (1.8), we then have

$$\frac{t}{\xi e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}\frac{t^n}{n!},$$

where $B_{n,\xi}$ denotes the classical twisted Bernoulli numbers (cf. [9,22,23,31]; see also the references cited in each of these earlier works).

If $q \to 1$ in (1.10), then we have

$$B_{n,\xi} = \frac{nH_{n-1}(\xi^{-1})}{\xi - 1}, \quad n \geq 1 \text{ (cf. [9])}.$$

The twisted $(h, q)$-extensions of Bernoulli polynomials $B^{(h)}_{n,\xi}(z, q)$ are defined by means of the following generating function [33]:

Let

$$\mathcal{F}^{(h)}_{\xi,q}(t, z) = F^{(h)}_{\xi,q}(t)e^{tz}.$$

From the above, we have

$$\mathcal{F}^{(h)}_{\xi,q}(t, z) = \frac{(t + \log q^h)e^{tz}}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B^{(h)}_{n,\xi}(z, q)\frac{t^n}{n!}, \quad |t + \log(\xi q^h)| < 2\pi, \tag{1.11}$$

where $\xi$ is an $r$th root of 1.

Observe that $B^{(h)}_{n,\xi}(0, q) = B^{(h)}_{n,\xi}(q)$.

By using (1.11), we have

$$B^{(h)}_{n,\xi}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} B^{(h)}_{k,\xi}(q) \tag{1.12}$$

(cf. [33]).

Substituting $\xi = 1$ into (1.11), we have

$$\frac{(t + \log q^h)e^{tz}}{q^h e^t - 1} = \sum_{n=0}^{\infty} B^{(h)}_n(z, q)\frac{t^n}{n!}, \quad |t + \log(q^h)| < 2\pi \text{ (cf. [16])},$$

and setting $\xi = 1$, $q \to 1$ in (1.11), we have

$$\frac{te^{tz}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z)\frac{t^n}{n!}, \quad |t| < 2\pi$$

where $B_n(x)$ denotes the usual Bernoulli polynomials (see [1–44]).

We summarize our paper as follows.

In Section 2, by applying the Mellin transformation to the generating functions for the Bernoulli polynomials and generalized Bernoulli polynomials, we give an integral representation of the twisted $(h, q)$-Hurwitz function and the twisted $(h, q)$-two-variable $L$-function. By using these functions, we construct a new twisted $(h, q)$-partial zeta function which interpolates the twisted $(h, q)$-Bernoulli polynomials at negative integers. We give a relation between the twisted $(h, q)$-partial zeta functions and the twisted $(h, q)$-two-variable $L$-function.

In Section 3, we construct a two-variable $p$-adic twisted $(h, q)$-$L$-function which interpolates the twisted generalized $(h, q)$-Bernoulli polynomials at negative integers. We calculate the residue of this function at $s = 1$. We also give some relations related to this function.

In Section 4, we give an integral representation of the $p$-adic twisted $(h, q)$-$L$-function.

## 2. $(h, q)$-partial zeta functions

Our primary aim in this section is to define twisted $(h, q)$-partial zeta functions. We give a relation between the generating function in (1.11) and the twisted $(h, q)$-Hurwitz zeta function [34]. Throughout this section, our study is for the complex $s$-plane with $\Re(s) > 2$. Let us have $q \in \mathbb{C}$ with $|q| < 1$ and $\xi^r = 1$ ($\xi$ is an $r$th root of 1), $\xi \neq 1$.

For $s \in \mathbb{C}$, by applying the Mellin transformation to (1.11), we have

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} \mathcal{F}^{(h)}_{\xi,q}(-t, x)dt = \zeta^{(h)}_{\xi,q}(s, x).$$

By using the above equation, we defined the twisted $(h, q)$-Hurwitz zeta function as follows (cf. [33,34]):

**Definition 1.** Let us have $s \in \mathbb{C}$, with $\Re(s) > 2$, $0 < x \leq 1$. We define

$$\zeta^{(h)}_{\xi,q}(s, x) = \sum_{n=0}^{\infty} \frac{\xi^n q^{nh}}{(n + x)^s} - \frac{h \log q}{s - 1} \sum_{n=0}^{\infty} \frac{\xi^n q^{nh}}{(n + x)^{s-1}}. \tag{2.1}$$

**Remark 1.** Substituting $x = 1$ into (2.1), we have

$$\zeta_{\xi,q}^{(h)}(s) = \sum_{n=1}^{\infty} \frac{\xi^n q^{nh}}{n^s} - \frac{h \log q}{s-1} \sum_{n=1}^{\infty} \frac{\xi^n q^{nh}}{n^{s-1}}, \tag{2.2}$$

which is analytically continued for Re $(s) > 2$ (cf. [33]). If $\xi = 1$, then (2.1) reduces to

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^s} - \frac{h \log q}{s-1} \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^{s-1}}, \quad \Re(s) > 2$$

(cf. [16]). If $q \to 1$, $\xi = 1$, then $\zeta_{q,\xi}^{(h)}(s)$ reduces to the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1$$

and $\zeta_{q,\xi}^{(h)}(s,x)$ reduces to the Hurwitz zeta function:

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \Re(s) > 1$$

(cf. [2,6,16,7,33,38,41]; see also the references cited in each of these earlier works).

Replacing $n$ by $j + yk$, with $j = 1, 2, \ldots, y$ and $k = 0, 1, \ldots, \infty$, in (2.2), after some elementary calculations, we arrive at the relation between $\zeta_{\xi,q}^{(h)}(s,x)$ and $\zeta_{\xi,q}^{(h)}(s)$ via the following corollary:

**Corollary 1.** *Let us have $y \in \mathbb{Z}^+$. Then we have*

$$\zeta_{\xi,q}^{(h)}(s) = y^{-s} \sum_{j=1}^{y} \xi^j q^{hj} \zeta_{\xi^y,q^y}^{(h)}\left(s, \frac{j}{y}\right).$$

Observe that substituting $q \to 1$, $\xi = 1$ into Corollary 1, we then have

$$\zeta(s) = y^{-s} \sum_{j=1}^{y} \zeta\left(s, \frac{j}{y}\right) \quad \text{(cf. [38])}.$$

The family of the twisted zeta functions is related to the *Lerch transcendent* $\Phi(z,s,x)$ (cf. [38,6]), which is the analytic continuation of the series

$$\Phi(z,s,x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s},$$

which converges for $\{x \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z| < 1\}$ or $\{\Re(s) > 1$ when $|z| = 1\}$ where

$$\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}, \quad \mathbb{Z}^- = \{-1, -2, -3, \ldots\}.$$

$\Phi$ denotes the familiar Hurwitz–Lerch zeta function (cf. [38, p. 121], [6]). The function $\Phi$ is related to the following special functions:

$$\zeta_{\xi,q}^{(h)}(s,x) = \Phi(\xi q^h, s, x) - \frac{h \log q}{s-1} \Phi(\xi q^h, s-1, x), \quad \Re(s) > 2 \tag{2.3}$$

$$\Phi(1, s, 1) = \zeta(s), \quad \Re(s) > 1$$

and

$$\Phi(1, s, x) = \zeta(s,x), \quad \Re(s) > 1$$

and the Dirichlet eta function $\eta(s)$:

$$\zeta_{-1,1}^{(1)}(s,1) = \Phi(-1, s, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) > 1$$

(cf. [2,6,38]; see also the references cited in each of these earlier works).

**Theorem 1** (*[34]*). *Let us have $n \in \mathbb{Z}^+$. We obtain*

$$\zeta_{\xi,q}^{(h)}(1-n,x) = -\frac{B_{n,\xi}^{(h)}(x,q)}{n}. \tag{2.4}$$

The twisted $(h,q)$-$L$-function is defined as follows:

**Definition 2** (*[33]*). Let us have $s \in \mathbb{C}$ with $\Re(s) > 2$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We define

$$L_{\xi,q}^{(h)}(s,\chi) = \sum_{n=1}^{\infty} \frac{q^{nh}\xi^n \chi(n)}{n^s} - \frac{\log q^h}{s-1} \sum_{n=1}^{\infty} \frac{q^{nh}\xi^n \chi(n)}{n^{s-1}}. \tag{2.5}$$

Observe that if $\xi = 1$, (2.5) reduces to

$$L_{q}^{(h)}(s,\chi) = \sum_{n=1}^{\infty} \frac{q^{nh} \chi(n)}{n^s} - \frac{\log q^h}{s-1} \sum_{n=1}^{\infty} \frac{q^{nh} \chi(n)}{n^{s-1}}$$

(cf. [16]). If $q \to 1$ and $\xi = 1$, (2.5) reduces to (1.1).

**Theorem 2** (*[33]*). *Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Let us have $n \in \mathbb{Z}^+$. Then we have*

$$L_{\xi,q}^{(h)}(-n,\chi) = -\frac{B_{n+1,\chi,\xi}^{(h)}(q)}{n+1},$$

*where $B_{n+1,\chi,\xi}^{(h)}(q)$ is defined by the following generating function:*

$$\sum_{a=1}^{f} \frac{\chi(a)\xi^a q^{ha} e^{at}(t+\log q^h)}{\xi^f q^{hf} e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}^{(h)}(q) \frac{t^n}{n!}, \quad |t + \log(\xi q^h)| < \frac{2\pi}{f}.$$

The relations between $\zeta_{w,q}^{(h)}(s,z)$ and $L_{w,q}^{(h)}(s,\chi)$ are given by the following theorem (cf. [33]):

**Theorem 3.** *Let us have $s \in \mathbb{C}$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We have*

$$L_{\xi,q}^{(h)}(s,\chi) = f^{-s} \sum_{a=0}^{f-1} q^{ha}\xi^a \chi(a) \zeta_{\xi^f,q^f}^{(h)}\left(s, \frac{a}{f}\right). \tag{2.6}$$

By (2.3) and (2.6), we obtain the following corollary:

**Corollary 2.**

$$L_{\xi,q}^{(h)}(s,\chi) = f^{-s} \sum_{a=0}^{f-1} q^{ha}\xi^a \chi(a) \left( \Phi\left(\xi^f q^{fh}, s, \frac{a}{f}\right) - \frac{h \log q^f}{s-1} \Phi\left(\xi^f q^{fh}, s-1, \frac{a}{f}\right)\right).$$

The generalized twisted $(h,q)$-extensions of the Bernoulli polynomials $B_{n,\chi,\xi}^{(h)}(z,q)$ are defined by means of the generating function (cf. [34])

$$\begin{aligned}
\mathfrak{F}_{\chi,w,q}^{(h)}(t,z) &= t \sum_{a=1}^{f} \frac{\chi(a)\xi^a q^{ha} e^{(z+a)t}}{\xi^f q^{hf} e^{ft} - 1} + \log q^h \sum_{a=1}^{f} \frac{\chi(a)\xi^a q^{ha} e^{(z+a)t}}{\xi^f q^{hf} e^{ft} - 1} \\
&= \sum_{a=1}^{f} \frac{\chi(a)\xi^a q^{ha} e^{(z+a)t}(t+\log q^h)}{\xi^f q^{hf} e^{ft} - 1} \\
&= \sum_{n=0}^{\infty} B_{n,\chi,\xi}^{(h)}(z,q) \frac{t^n}{n!}, \quad |t + \log(\xi q^h)| < \frac{2\pi}{f} \text{ (cf. [33,34])}.
\end{aligned} \tag{2.7}$$

By (2.7), we now give the following relation:

$$\mathfrak{F}_{\chi,w,q}^{(h)}(t,z) = -t \sum_{m=0}^{\infty} \chi(m)\xi^m q^{hm} e^{(z+m)t} - \log q^h \sum_{m=0}^{\infty} \chi(m)\xi^m q^{hm} e^{(z+m)t}.$$

The series on the right-hand side of the above equations is uniformly convergent. Thus we obtain

$$
\begin{aligned}
B^{(h)}_{k,\chi,\xi}(z,q) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathfrak{F}^{(h)}_{\chi,w,q}(t,z)\,|_{t=0} \\
&= -k\sum_{m=0}^{\infty}\chi(m)\xi^m q^{hm}(m+z)^{k-1} - \log q^h \sum_{m=0}^{\infty}\chi(m)\xi^m q^{hm}(m+z)^k.
\end{aligned}
$$

After some elementary calculations, we have

$$
\sum_{m=0}^{\infty}\chi(m)\xi^m q^{hm}(m+z)^{k-1} + \frac{\log q^h}{k}\sum_{m=0}^{\infty}\chi(m)\xi^m q^{hm}(m+z)^k = -\frac{B^{(h)}_{k,\chi,\xi}(z,q)}{k}, \tag{2.8}
$$

where

$$
B^{(h)}_{n,\chi,\xi}(z,q) = \sum_{k=0}^{n}\binom{n}{k} z^{n-k} B^{(h)}_{k,\chi,\xi}(q)
$$

(cf. [33,34]). For any positive integer $n$, we have

$$
B^{(h)}_{n,\chi,\xi}(z,q) = f^{n-1}\sum_{a=1}^{f}\chi(a)\xi^a q^{ha} B^{(h)}_{n,\xi^f}\left(\frac{a+z}{f}, q^f\right). \tag{2.9}
$$

Note that $B^{(h)}_{n,\chi,\xi}(0,q) = B^{(h)}_{n,\chi,\xi}(q)$, $\lim_{q\to 1} B^{(h)}_{n,\chi,\xi}(q) = B^{(h)}_{n,\chi,\xi}$, where $B^{(h)}_{n,\chi,\xi}$ are the twisted Bernoulli numbers (cf. [33]). In [9], Kim studied analogues of the Bernoulli numbers, called twisted Bernoulli numbers in this paper. He proved a relation between these numbers and Frobenius–Euler numbers. If $\xi = 1$ and $q \to 1$, then $B_{n,\chi,\xi}(q) \to B_{n,\chi}$ are the usual generalized Bernoulli numbers, and $B_{n,\chi,\xi}(z,q) \to B_{n,\chi}(z)$ are the usual generalized Bernoulli polynomials; cf. [37]. Observe that $\mathcal{F}^{(h)}_{\xi,q}(t,z)\,|_{\xi=1} = F^{(h)}_q(t,z)$; cf. [16].

By (2.7), we define a twisted two-variable $(h,q)$-$L$-function. For $s \in \mathbb{C}$ with $\Re(s) > 2$, we consider the integral below, which is known as the Mellin transformation of $\mathfrak{F}^{(h)}_{\chi,\xi,q}(t,z)$ (cf. [34]):

$$
\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2}\mathfrak{F}^{(h)}_{\chi,\xi,q}(-t,z)\mathrm{d}t = L^{(h)}_{\xi,q}(s,z,\chi). \tag{2.10}
$$

We are now ready to define the new twisted two-variable $(h,q)$-$L$ function. By using the above integral representation we arrive at the following definition:

**Definition 3** (*[34]*). Let us have $s \in \mathbb{C}$ with $\Re(s) > 2$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Let $\xi^r = 1, \xi \neq 1$. We define

$$
L^{(h)}_{\xi,q}(s,z,\chi) = \sum_{m=0}^{\infty}\frac{\chi(m)\xi^m q^{hm}}{(z+m)^s} - \frac{\log q^h}{s-1}\sum_{m=0}^{\infty}\frac{\chi(m)\xi^m q^{hm}}{(z+m)^{s-1}}. \tag{2.11}
$$

The relation between $\zeta^{(h)}_{\xi,q}(s,z)$ and $L^{(h)}_{\xi,q}(s,z,\chi)$ is given by the following theorem:

**Theorem 4** (*[34]*). *We have*

$$
L^{(h)}_{\xi,q}(s,z,\chi) = f^{-s}\sum_{a=1}^{f} q^{ha}\xi^a\chi(a)\zeta^{(h)}_{\xi^f,q^f}\left(s, \frac{a+z}{f}\right). \tag{2.12}
$$

**Theorem 5.** *Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$. We have*

$$
L^{(h)}_{\xi,q}(1-n,z,\chi) = -\frac{B^{(h)}_{n,\chi,\xi}(z,q)}{n}. \tag{2.13}
$$

**Proof.** Substituting $s = 1-n$, $n \in \mathbb{Z}^+$, into (2.11), we obtain

$$
L^{(h)}_{\xi,q}(1-n,z,\chi) = \sum_{m=0}^{\infty}\frac{\chi(m)\xi^m q^{hm}}{(z+m)^{1-n}} + \frac{\log q^h}{n}\sum_{m=0}^{\infty}\frac{\chi(m)\xi^m q^{hm}}{(z+m)^{-n}}.
$$

Substituting (2.8) into the above equation, we arrive at the desired result. □

**Remark 2.** The proof of (2.13) runs parallel to that of Theorem 8 in [37]; for $s = 1 - n$, $n \in \mathbb{Z}^+$, and by using the Cauchy Residue Theorem for (2.10), we arrive at another proof of the above theorem [34]. Observe that if $\xi = 1$, then we have $L_{\xi,q}^{(h)}(s, 1, \chi) = L_q^{(h)}(s, \chi)$. For $q \to 1$ and $z = 1$, the relation (2.11) reduces to the following well-known definition:

Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$, and let $\xi^r = 1$, $\xi \neq 1$. The twisted $L$-functions are defined by [23]

$$L_\xi(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\xi^n}{n^s}.$$

Since the function $n \to \chi(n)\xi^n$ has period $fr$, this is a special case of the Dirichlet $L$-function. Koblitz [23] and the author gave a relation between $L_\xi(s, \chi)$ and the twisted Bernoulli numbers, $B_{n,\chi,\xi}$, at non-positive integers (cf. [22,23,31,30]; see also the references cited in each of these earlier works).

Let $s$ be a complex variable, $a$ and $F$ be integers with $0 < a < F$. Then we define a new twisted $(h, q)$-partial zeta function as follows:

**Definition 4.** Let us have $s \in \mathbb{C}$ with $\Re(s) > 2$. Let $\xi^r = 1$, $\xi \neq 1$. We define

$$H_{\xi,q}^{(h)}(s, a : F) = \sum_{\substack{n \equiv a (\mathrm{mod} F)\\ n>0}}^{\infty} \frac{q^{nh}\xi^n}{n^s} - \frac{\log q^h}{s-1} \sum_{\substack{n \equiv a (\mathrm{mod}\ F)\\ n>0}}^{\infty} \frac{q^{nh}\xi^n}{n^{s-1}}.$$

By using the above definition, the relations between $H_{\xi,q}^{(h)}(s, a : F)$ and $\zeta_{\xi,q}^{(h)}(s, x)$ are given by the following theorem:

**Theorem 6.** *Let us have $s \in \mathbb{C}$ and $\xi^r = 1$, $\xi \neq 1$. Then we have*

$$H_{\xi,q}^{(h)}(s, a : F) = q^{ha}\xi^a F^{-s} \zeta_{\xi^F, q^F}^{(h)}\left(s, \frac{a}{F}\right). \tag{2.14}$$

**Remark 3.** The function $H_{\xi,q}^{(h)}(s, a : F)$ is a meromorphic function for $s \in \mathbb{C}$ with a simple pole at $s = 1$, having the residue $\mathrm{Re}\, z_{s=1} H_{\xi,q}^{(h)}(s, a : F)$:

$$\begin{aligned}\mathrm{Re}\, z_{s=1} H_{\xi,q}^{(h)}(s, a{:}\, F) &= \lim_{s\to 1}(s-1) H_{\xi,q}^{(h)}(s, a : F)\\ &= \frac{q^{ha}\xi^a \log q^h}{q^{hF}\xi^F - 1}.\end{aligned}$$

By using (2.3) and (2.14), we arrive at the following corollary:

**Corollary 3.**

$$H_{\xi,q}^{(h)}(s, a : F) = q^{ha}\xi^a F^{-s}\left(\Phi\left(\xi^F q^{Fh}, s, \frac{a}{F}\right) - \frac{h \log q^F}{s-1}\Phi\left(\xi^F q^{Fh}, s-1, \frac{a}{F}\right)\right).$$

By using (2.4) and (2.14), we arrive at the following corollary:

**Corollary 4.** *Let us have $n \in \mathbb{Z}^+$ and $\xi^r = 1$, $\xi \neq 1$. Then we have*

$$H_{\xi,q}^{(h)}(1-n, a : F) = -\frac{q^{ha}\xi^a F^{n-1} B_{n,\xi^F}^{(h)}\left(\frac{a}{F}, q^F\right)}{n}. \tag{2.15}$$

We modify the twisted $(h, q)$-extension of the partial zeta function as follows:

**Corollary 5.** *Let us have $s \in \mathbb{C}$ and $\xi^r = 1$, $\xi \neq 1$. Then we have*

$$H_{\xi,q}^{(h)}(s, a : F) = \frac{a^{s-1}q^{ha}\xi^a}{(s-1)F} \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a}\right)^k B_{k,\xi^F}^{(h)}(q^F). \tag{2.16}$$

**Proof.** By using (1.12) and (2.15), we have

$$H_{\xi,q}^{(h)}(1-n, a : F) = -\frac{q^{ha}\xi^a F^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{F}\right)^{n-k} B_{k,\xi^F}^{(h)}\left(\frac{a}{F}, q^F\right).$$

Substituting $s = 1 - n$, and after some elementary calculations, we arrive at the desired result. □

Observe that if $\xi = 1$, then $H_q^{(h)}(s, a : F)$ is reduced to the following equation (cf. [18]):

$$H_q^{(h)}(s, a : F) = \frac{a^{s-1}q^{ha}}{(s-1)F} \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a}\right)^k B_k^{(h)}(q^F).$$

By using (2.6), (2.14) and (2.16), we arrive at the following theorem:

**Theorem 7.** *Let us have $s \in \mathbb{C}$ and $\xi^r = 1, \xi \neq 1$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$ and $F$ be any multiple of $f$. We have*

$$\begin{aligned} L_{\xi,q}^{(h)}(s, \chi) &= \sum_{a=1}^{F} \chi(a) H_{\xi,q}^{(h)}(s, a : F) \\ &= \frac{1}{(s-1)F} \sum_{a=1}^{F} \chi(a) a^{s-1} q^{ha} \xi^a \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a}\right)^k B_{k,\xi^F}^{(h)}(q^F). \end{aligned}$$

We now define a new twisted $(h, q)$-partial Hurwitz zeta function as follows:

**Definition 5.** Let us have $s \in \mathbb{C}$ with $\Re(s) > 2$ and $\xi^r = 1, \xi \neq 1$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We define

$$H_{\xi,q}^{(h)}(s, x+a : F) = \sum_{\substack{n \equiv a (\mathrm{mod} F) \\ n \geq 0}}^{\infty} \frac{q^{nh}\xi^n}{(x+n)^s} - \frac{\log q^h}{s-1} \sum_{\substack{n \equiv a (\mathrm{mod} F) \\ n \geq 0}}^{\infty} \frac{q^{nh}\xi^n}{(x+n)^{s-1}}.$$

The relations between $\zeta_{\xi,q}^{(h)}(s, x)$ and $H_{\xi,q}^{(h)}(s, x+a : F)$ are given by

$$H_{\xi,q}^{(h)}(s, x+a : F) = \frac{q^{ha}\xi^a}{F^s} \zeta_{\xi^F,q^F}^{(h)}\left(s, \frac{a+x}{F}\right). \tag{2.17}$$

Let us have $n \in \mathbb{Z}^+$. Substituting (2.4) in the above and using (1.12), we obtain

$$\begin{aligned} H_{\xi,q}^{(h)}(1-n, x+a : F) &= -\frac{B_{n,\xi^F}^{(h)}\left(\frac{a+x}{F}, q^F\right)}{n} \\ &= -\frac{q^{ha}\xi^a F^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x+a}{F}\right)^{n-k} B_{k,\xi^F}^{(h)}\left(\frac{x+a}{F}, q^F\right). \end{aligned}$$

Thus, by the above equation, we obtain

$$H_{\xi,q}^{(h)}(s, x+a : F) = \frac{a^{1-s}q^{ha}\xi^a}{(s-1)F} \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{x+a}\right)^k B_{k,\xi^F}^{(h)}(q^F).$$

By (2.12) and (2.17), we obtain the following relations:

Let us have $s \in \mathbb{C}$. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$ and $F$ be any multiple of $f$. We have

$$\begin{aligned} L_{\xi,q}^{(h)}(s, x, \chi) &= \sum_{a=1}^{F} \chi(a) H_{\xi,q}^{(h)}(s, x+a : F) \\ &= \frac{1}{(s-1)F} \sum_{a=1}^{F} \chi(a)(x+a)^{1-s} q^{ha}\xi^a \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{x+a}\right)^k B_{k,\xi^F}^{(h)}(q^F). \end{aligned} \tag{2.18}$$

By the above equation, $L_{\xi,q}^{(h)}(s, x, \chi)$ is analytic for $x \in \mathbb{R}$ with $0 < x < 1$ and we have that $s \in \mathbb{C}$ except for $s = 1$.

**Remark 4.** Observe that if $\xi = 1$, then $L_{\xi,q}^{(h)}(s, x, \chi)$ is reduced to the following equation (cf. [18]):

$$L_q^{(h)}(s, x, \chi) = \frac{1}{(s-1)F} \sum_{a=1}^{F} \chi(a)(x+a)^{1-s} q^{ha} \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{x+a}\right)^k B_k^{(h)}(q^F).$$

By (2.13), the values of $L_{\xi,q}^{(h)}(s, x, \chi)$ at negative integers are algebraic, and hence may be regarded as lying in an extension of $\mathbb{Q}_p$. Consequently, we investigate a $p$-adic function which agrees with negative integers in the next section.

Substituting $s = 0$ into (2.18), we obtain

$$L_{\xi,q}^{(h)}(0, x, \chi) = \sum_{a=1}^{F} \chi(a)(x+a) q^{ha}\xi^a \left(\frac{F\xi^F q^{Fh} \log q^{Fh} - (x+a)(\xi^F q^{Fh} - 1) \log q^{Fh}}{F(x+a)(\xi^F q^{Fh} - 1)^2}\right).$$

## 3. The twisted *p*-adic interpolation function for the *q*-extension of the generalized Bernoulli polynomials

In this section, we construct a two-variable $p$-adic twisted $(h, q)$-$L$-function. This function interpolates generalized $(h, q)$-Bernoulli numbers at negative integers. We calculate the residue of this function at $s = 1$. Finally, we give some relations related to this function.

Here, we can use some notation due to Washington [41], Koblitz [22] and Kim [18]. Let $w$ denote the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character $\chi$, we define $\chi_n = \chi w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. If $q \in \mathbb{C}_p$, then we assume that $|1-q|_p < p^{-\frac{1}{p-1}}$. Let $\langle a\rangle = w^{-1}(a)a = \frac{a}{w(a)}$. We note that $\langle a\rangle \equiv 1(\mathrm{mod}\ p^*\mathbb{Z}_p)$. Thus, we see that

$$\begin{aligned}\langle a+p^*t\rangle &= w^{-1}(a+p^*t)(a+p^*t)\\ &= w^{-1}(a)a + w^{-1}(a)(p^*t) \equiv 1(\mathrm{mod} p^*\mathbb{Z}_p[t]),\end{aligned}$$

where $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, $(a, p) = 1$.

Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j}x^n,$$

where $a_{n,j} \in \mathbb{C}_p$, $j = 0, 1, 2, \ldots$, be a sequence of power series, each of which converges in $\mathcal{D} \subset \mathbb{C}_p$ such that:

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$, for $\forall n$,

(2) for each $s \in \mathcal{D}$ and $\epsilon > 0$, there exists $n_0 = n_0(s, \epsilon)$ such that $|\sum_{n\geq n_0} a_{n,j}s^n|_p < \epsilon$ for $\forall j$.

Then $\lim_{j\to\infty} A_j(s) = A_0(s)$ for all $s \in \mathcal{D}$. According to Washington [41], the functions $w^{-s}(a)a^s$ and

$$\sum_{k=0}^{\infty}\binom{s}{k}\left(\frac{F}{a}\right)^k B_k \quad (\text{cf. } [41])$$

are analytic in $\mathcal{D}$.

We construct twisted $p$-adic analogues of the twisted two-variable $q$-$L$-function. Let $F$ be a positive integral multiple of $p^*$ and $f = f_\chi$.

We define

$$L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle^{1-s}q^{ha}\xi^a\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{a+p^*t}\right)^k B^{(h)}_{k,\xi^F}(q^F), \tag{3.1}$$

where $\xi \in \mathbb{T}_p$, and $\chi$ is a Dirichlet character of conductor $f$, and let $F$ be any multiple of $p^*$ and $f$. Then $L^{(h)}_{\xi,p,q}(s,t,\chi)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, provided $s \in \mathcal{D}$, except for $s = 1$ when $\chi \neq 1$. For $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, we see that

$$\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{a+p^*t}\right)^k B^{(h)}_{k,\xi^F}(q^F)$$

is analytic for $s \in \mathcal{D}$. By the definition of $\langle a+p^*t\rangle$, we easily see that

$$\langle a+p^*t\rangle^s = \langle a\rangle^s\sum_{k=0}^{\infty}\binom{s}{k}(a^{-1}p^*t)^k$$

is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$ when $s \in \mathcal{D}$. Since $(s-1)L^{(h)}_{\xi,p,q}(s,t,\chi)$ is a finite sum of products of these two functions, it must also be analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, whenever $s \in \mathcal{D}$.

Observe that

$$\lim_{s\to 1}(s-1)L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)q^{ha}\xi^a B^{(h)}_{0,\xi^F}(q^F).$$

Substituting $\chi \equiv 1$ in the above, we then have

$$\begin{aligned}\lim_{s\to 1}(s-1)L^{(h)}_{\xi,p,q}(s,t,\chi) &= \frac{B^{(h)}_{0,\xi^F}(q^F)}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}q^{ha}\xi^a\\ &= \frac{B^{(h)}_{0,\xi^F}(q^F)}{F}\left(\frac{1-q^{hF}\xi^F}{1-q^h\xi}-\frac{1-q^{hpF}}{1-q^{ph}}\right).\end{aligned}$$

By the definition of $B^{(h)}_{0,\xi^F}(q^F)$ in (1.9), we obtain

$$\begin{aligned}\operatorname{Re} z_{s=1} L^{(h)}_{\xi,p,q}(s,t,\chi) &= \lim_{s\to 1}(s-1)L^{(h)}_{\xi,p,q}(s,t,\chi)\\ &= \frac{\log q^h}{q^h\xi - 1}\left(\frac{1-q^{hF}\xi^F}{1-q^h\xi} - \frac{1-q^{hpF}}{1-q^{hp}}\right),\end{aligned}$$

when $\chi \equiv 1$. Let us have $n \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$. Since $F$ must be a multiple of $f = f_{\chi_n}$, by (2.9), we obtain

$$B^{(h)}_{n,\chi_n,\xi}(p^*t, q) = F^{n-1}\sum_{a=0}^{F}\chi_n(a)\xi^a q^{ha} B^{(h)}_{n,\xi^F}\left(\frac{a+p^*t}{F}, q^F\right). \tag{3.2}$$

If $\chi_n(p) = 0$, then $(p, f_{\chi_n}) = 1$, so $\frac{F}{p}$ is a multiple of $f_{\chi_n}$. Consequently, we get

$$\chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t, q^p) = F^{n-1}\sum_{\substack{a=1\\ p\mid a}}^{F}\chi_n(a)\xi^a q^{ha} B^{(h)}_{n,\xi^F}\left(\frac{a+p^*t}{F}, q^F\right). \tag{3.3}$$

Taking the difference of (3.2) and (3.3), we have

$$B^{(h)}_{n,\chi_n,\xi}(p^*t, q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t, q^p) = F^{n-1}\sum_{\substack{a=0\\ p\nmid a}}^{F}\chi_n(a)\xi^a q^{ha} B^{(h)}_{n,\xi^F}\left(\frac{a+p^*t}{F}, q^F\right).$$

By using (1.12), we obtain

$$\begin{aligned}B^{(h)}_{n,\xi^F}\left(\frac{a+p^*t}{F}, q^F\right) &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{a+p^*t}{F}\right)^{n-k} B^{(h)}_{k,\xi^F}(q^F)\\ &= (a+p^*t)^n F^{-n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a+p^*t}\right)^{k} B^{(h)}_{k,\xi^F}(q^F).\end{aligned}$$

Since $\chi_n(a) = \chi(a)w^{-n}(a)$, $(a,p) = 1$, and $t \in \mathbb{C}_p$ with $|t|_p \le 1$, we have

$$B^{(h)}_{n,\chi_n,\xi}(p^*t, q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t, q^p) = \frac{1}{F}\sum_{\substack{a=1\\ (a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle^n q^{ha}\xi^a \sum_{k=0}^{\infty}\binom{n}{k}\left(\frac{F}{a+p^*t}\right)^{k} B^{(h)}_{k,\xi^F}(q^F).$$

Substituting $s = 1-n$, $n \in \mathbb{Z}^+$, into (3.1), we obtain

$$L^{(h)}_{\xi,p,q}(1-n,t,\chi) = -\frac{B^{(h)}_{n,\chi_n,\xi}(p^*t, q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t, q^p)}{n}.$$

Consequently, we arrive at the following main theorem:

**Theorem 8.** *Let us have $\xi \in \mathbb{T}_p$. Let $\chi$ be a Dirichlet character of conductor $f$ and $F$ be any multiple of $p^*$ and $f$. Let us have $s \in \mathcal{D}$. Then we have*

$$L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)F}\sum_{\substack{a=1\\ (a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle^{1-s} q^{ha}\xi^a \sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{a+p^*t}\right)^{k} B^{(h)}_{k,\xi^F}(q^F).$$

*Then $L^{(h)}_{\xi,p,q}(s,t,\chi)$ is analytic for $h \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$, provided $s \in \mathcal{D}$, except for $s = 1$. Also, if $t \in \mathbb{C}_p$ with $|t|_p \le 1$, this function is analytic for $s \in \mathcal{D}$ when $\chi \ne 1$, and meromorphic for $s \in \mathcal{D}$, with a simple pole at $s = 1$ having residue*

$$\frac{\log q^h}{q^h\xi - 1}\left(\frac{1-q^{hF}\xi^F}{1-q^h\xi} - \frac{1-q^{hpF}}{1-q^{ph}}\right)$$

*when $\chi = 1$. In addition, for each $n \in \mathbb{Z}^+$, we have*

$$L^{(h)}_{\xi,p,q}(1-n,t,\chi) = -\frac{B^{(h)}_{n,\chi_n,\xi}(p^*t, q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t, q^p)}{n}.$$

**Remark 5.** Observe that if $\xi = 1$, then $L^{(h)}_{1,p,q}(s,t,\chi) = L^{(h)}_{p,q}(s,t,\chi)$ (cf. [18]). If $h = 1$, $L^{(1)}_{p,q}(s,0,\chi) = L_{p,q}(s,\chi)$ (cf. [10,12]). $\lim_{q\to 1} L_{p,q}(s,\chi) = L_p(s,\chi)$ (cf. [3,4,7,22,23,5,28,41,35]).

## 4. Applications

In this section, we give some applications related to the twisted $p$-adic interpolation function for the $q$-extension of the generalized Bernoulli polynomials.

The Witt type formula for the $B^{(h)}_{n,\chi,\xi}(z, q)$ polynomials is defined as follows:

$$B^{(h)}_{n,\chi,\xi}(q) = \int_{\mathbb{X}} \chi(x)\phi_\xi(x)q^{hx}x^n \mathrm{d}\mu_1(x) \quad \text{(cf. [33,34])}$$

where $|1-q|_p \leq p^{-\frac{1}{p-1}}$.

Let

$$Y = \left\{q \in \mathbb{C}_p : |q-1| < 1\right\}$$

and let $\overline{Y} = \mathbb{C}_p \setminus Y$ be the complement of the open unit disc around 1. According to Kim [11], if $q \in \overline{Y}$ and $\mathrm{ord}_p(1-q) \neq -\infty$, then $\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$ is the measure. We assume that $q \in \overline{Y}$ and $\mathrm{ord}_p(1-q) \neq -\infty$ (see also [21]).

By using the above formula and the $p$-adic $q$-Volkenborn integral, we modify the twisted $p$-adic interpolation function as follows:

$$L^{(h)}_{\xi,p,q}(s, \chi) = \frac{1}{s-1}\int_{\mathbb{X}^*} \chi(x)\phi_\xi(x)\langle x\rangle^{-s}q^{hx}\mathrm{d}\mu_q(x),$$

where $\xi \in \mathbb{T}_p$ and $q \in \overline{Y}$, with $\mathrm{ord}_p(1-q) \neq -\infty$, and letting $\chi$ be a Dirichlet character of conductor $f$ and $F$ be any multiple of $p^*$ and $f$, and having that $s \in \mathcal{D}$.

Substituting $s = 1-n$, $n \in \mathbb{Z}^+$, into the above, after some calculations, we obtain

$$\begin{aligned} L^{(h)}_{\xi,p,q}(1-n, \chi) &= -\frac{1}{n}\int_{\mathbb{X}^*} \phi_\xi(x)\chi(x)\langle x\rangle^{n-1}q^{hx}\mathrm{d}\mu_q(x) \\ &= -\frac{1}{n}\left(\int_{\mathbb{X}} \chi_n(x)\phi_\xi(x)q^{hx}x^n\mathrm{d}\mu_q(x) - \int_{p\mathbb{X}} \chi_n(px)\phi_\xi(px)q^{phx}x^n\mathrm{d}\mu_{q^p}(x)\right) \\ &= -\frac{B^{(h)}_{n,\chi,\xi}(q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi,1}(q^p)}{n}. \end{aligned}$$

Consequently, we arrive at the following theorem:

**Theorem 9.** *Let us have $\xi \in \mathbb{T}_p$ and $q \in \overline{Y}$, with $\mathrm{ord}_p(1-q) \neq -\infty$. Let $\chi$ be a Dirichlet character of conductor $f$ and $F$ be any multiple of $p^*$ and $f$. Let us have $s \in \mathcal{D}$; we then have*

$$L^{(h)}_{\xi,p,q}(s, \chi) = \frac{1}{s-1}\int_{\mathbb{X}^*} \chi(x)\phi_\xi(x)\langle x\rangle^{-s}q^{hx}\mathrm{d}\mu_q(x).$$

*If $n \in \mathbb{Z}^+$, we have*

$$\begin{aligned} L^{(h)}_{\xi,p,q}(1-n, \chi) &= -\frac{1}{n}\int_{\mathbb{X}^*} \phi_\xi(x)\chi(x)\langle x\rangle^{n-1}q^{hx}\mathrm{d}\mu_q(x) \\ &= -\frac{B^{(h)}_{n,\chi,\xi}(q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi,1}(q^p)}{n}. \end{aligned}$$

**Remark 6.** By using the function $L^{(h)}_{\xi,p,q}(s, t, \chi)$, Kummer congruence of the generalized $(h, q)$-twisted Bernoulli numbers can be obtained. In this paper, we do not study Kummer congruence of the generalized $(h, q)$-twisted Bernoulli numbers.

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